CLOCKWORK SOLUTIONS

TECHNICAL PAPER

# The Monte Carlo Method vs Analytical Methods

In the world of Reliability Logistics and Systems Engineering, analytical methods for solving problems have been available and widely used for over 40 years. When Professor Arie Dubi was one of the first to introduce the MonteCarlo approach to such problems, analytical methods were, in fact, the exclusive tools in use. In Clockwork Solutions' extensive experience in predicting the behavior of industrial systems, we have come to appreciate the superiority of MonteCarlo methods. In this brief paper, Professor Dubi explains the differences between the two approaches and why MonteCarlo methods can simulate reality better and can lead to more accurate predictions than analytical methods in complex systems. We do not denigrate the continued use of analytical methods; we explain where they yield good results, where they fall short, and where MonteCarlo methods should, even must, be used.

**Example 1.** I have a pump. I have data. The failure distribution of the pump is a Weibull distribution, $f(t) = \lambda\beta t^{\beta-1} e^{-\lambda t^{\beta}}$. The values of $\lambda$ and $\beta$ (the scale and shape parameters) are known. I want to know the reliability of the pump at 1000 hours. The probability that the pump will not fail during the first 1000 hours of operation is $R(1000) = e^{-\lambda 1000^{\beta}}$.

**Example 2.** I know that the pump is in steady state. I also know that it takes, on the average, eight hours to repair the pump when it fails. I want to know the pump's availability, i.e., the probability that the pump is functioning at any given point in time. I also want to know how many failures should be expected in the next year. The pump's availability can be solved as follows:

$$T_f = \frac{\Gamma(1+\frac{1}{\beta})}{\lambda^{\frac{1}{\beta}}}$$ where $\Gamma(x)$ is the Gamma function, and the steady state availability is

$$A = \frac{T_f}{T_f + 8}$$

The average number of expected failures per year is

$$N = \frac{8760 - 8xN}{T_f} \Rightarrow N = \frac{8760}{T_f + 8}$$

The above formulas and calculations are in the realm of analytical methods. It is unnecessary to employ MonteCarlo methods to answer these questions. The next example shows how the MonteCarlo method enters the picture.


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**Example 3.** I want to know the availability of the pump for each year since it was bought new. At the end of each year, the pump undergoes an overhaul and thus never reaches steady state. What is the availability of the pump as a function of time for this year?

Here is the analytical solution:

$$A(t) = \int_0^t \psi(t')e^{-\lambda t'^{\beta}}dt' \text{ where } \psi(t') \text{ is the solution of the integral equation:}$$

$$\psi(t) = \delta(t) + \int_0^t \psi(t')\left[\int_{t'}^{t} \lambda\beta(t'-x)^{\beta-1}e^{-\lambda(t'-x)^{\beta}}g(x)dx\right]dt' \qquad (\delta(t) \text{ is an impulse function})$$

No known solution exists! Not by any analytical or deterministic means. However, with the MonteCarlo method, this is a trivial problem to solve.

## Mathematical Complexity

For Examples 1 and 2, analytical methods provide the answer. For Example 3, the MonteCarlo (MC) method is not merely a better option, it is the only option.

The above examples present the true dilemma of using MC or not. When presented with the question of the availability of the pump, one still has a choice to make. It is not a choice of which method to use. It is the choice of which question to ask. If one chooses to ask about the steady-state availability, analytical methods can and should be used. If one chooses to ask about the time-dependent availability, the MC method must be used.

How do we know which question to ask? When dealing with systems reliability logistics and engineering, the correct questions must be dictated by the reality of the system.

One may replace the question asked in Example 3 by that of Example 1, i.e., obtain the steady-state availability instead of the time-dependent availability arguing that the difference is not large and that the steady-state availability is a close enough approximation. This, however, is illogical because if one cannot obtain the time-dependent availability how does one know how accurate the approximation is?

The difficulty in solving the equation in Example 3 is representative of a large class of problems. It is prohibitively difficult to obtain an analytical solution**.** The MonteCarlo method is insensitive to this difficulty.

In the MC method, the difficulty of the integration of the equation (which, as in the solution of integral equations in general, is equivalent to a summation of an infinite series of multiple integrals–the Neumann series) is replaced by a detailed statistical follow-up of the random process leading to that equation.

The MC method was developed mainly in connection with the early development of nuclear technology. It was then necessary to solve a six-dimensional integral equation, not much unlike the equation in *Example 3*. There is only one simple case of that equation that has an analytical solution--infinite homogeneous medium with isotropic mono-energetic point source. One- and two-dimensional models can be solved by analytical methods. Six-dimensional solutions can be obtained only by using the MC method.


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Thus our first point is that the MC method overcomes the mathematical difficulty that prohibits the solution of complex problems. That difficulty limits analytical methods from being applied to a large and vital class of models such as:

1. Time-dependent availability, number of failures, or any other time-dependent properties (cost, throughput, etc.) except for the case in which all the distributions involved are exponential, i.e., constant failure and repair rates. This in turn means that no aging can be assumed. But how many realistic systems do not age?
2. No considerations of partial or minimum repair. Age reduction and therefore preventive maintenance cannot be analyzed. Performance as a function of partial repair and periodic or age-dependent maintenance cannot be studied.
3. Spare parts considerations, resource considerations, and any other logistic subjects that are time-dependent.

In short, the mathematical complexity (even for a single component) permits analytical solutions only if we assume that the failure and repair distributions (and any other distribution relevant to the situation such as recycling time) are exponential, or if we assume steady-state which eliminates time dependency altogether. However, a component that has an exponential distribution is not aging and all considerations of preventive maintenance are irrelevant for it.

Each of the above three cases is easily handled by the MonteCarlo method since it is insensitive to the mathematical complexity.

## Dimensionality

Mathematical complexity is not the only limitation of the analytical approach. A bigger problem arises from a different consideration - that of the dimensionality of the problem.

Multidimensional situations represent a profound difficulty in analytical methods. Consider the numerical integration of a one-dimensional function:

$$\int_a^b f(x)dx$$

This integral can be presented numerically as a simple sum:

$\sum_{i=1}^{n} f(x_i)\Delta x_i$ where the integration range is divided into n discrete intervals.

The error, $R_n$, involved in the approximation, is of the order of $R_n \propto \frac{1}{n}$, i.e., it is proportional to the reciprocal of the number of intervals used in the calculation. This rule pertains to any dimension.

In the one-dimensional case, the number of intervals used and the number of points summed are the same. If we look now at a three-dimensional function

$$\int_e^f \int_c^d \int_a^b f(x, y, z)dxdydz$$

and divide each coordinate into n intervals as we did for the one-dimensional case, the numerical approximation will take the form:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} f(x_i, y_j, z_k)\Delta x_i \Delta y_j \Delta z_k \ .$$

Again the error is $R_n \propto \frac{1}{n}$, and is proportional to the reciprocal of the number of intervals, but the number of points used in the sum is $M = n^3$ or $R_n \propto \frac{1}{M^{\frac{1}{3}}}$.

Thus to obtain the same error that was obtained with 100 intervals and 100 points in a one-dimensional calculation, we must use 1,000,000 points in a three-dimensional calculation, and, in general, $\mathbf{M = n^k}$ points in a k-dimensional calculation.

A 30-dimensional calculation with ten intervals on each axis will require $10^{30}$ points, which is far beyond the capacity of any computer.

## How do dimensions apply to systems calculations?

In a system calculation, each component represents a dimension. Thus a system with k components requires a k-dimensional calculation. Each possible state of a component is equivalent to an interval, thus the number of points to be considered is $\mathbf{M = n^k}$ where n is the number of component states and k is the number of components. Thus a system with 30 components each having three possible states, e.g., active, passive, and failed (in most realistic systems, components occur in more than just these three states) will have a total of $2{,}058 x 10^{14}$ points. It is this effect that profoundly limits the ability of analytical methods to ever be a solution for advanced calculations of modern complex system predictions.

It is the Central Limit Theorem that brings about MC's insensitivity to the mathematical complexity and dimensional complexity. The MonteCarlo method is thoroughly based on the Central Limit Theorem and thus does not require limiting approximations. There is no need to assume independence, constant failure rates, serial systems, etc. There is no need to separate spare parts from system throughput, from maintenance and partial repair, from limited number of repair teams and repair stations, from queuing and multiple dynamic storages, inspections etc. One can define the reality of the system to any degree of complexity, and the MonteCarlo method can deal with it.

## How Analytical Methods Have Been Employed

Despite their limitations, we have been using analytical methods for the past forty years. How can this be?

There is a way to deal with mathematical complexity and multi-dimensional limitations with analytical methods. It involves simplifications. Simplifications were used for a long time because nothing better was available. Thus they became standard practice.

In mathematics, it is common to try to replace a k-dimensional function by a product of k one-dimensional functions, namely, replace $f(x_1, x_2, x_3, \ldots\ldots, x_k)$ by $f_1(x_1) x f_2(x_2) x f_3(x_3) x \ldots\ldots x f_k(x_k)$ .

This has a dramatic effect on the degree of complexity of the problem. To the extent that the number of states can be a fair measure of complexity, this process reduces the complexity from $\mathbf{M = n^k}$ to $\mathbf{n x k}$ .

To see how dramatic this effect is, consider thirty components with three states each. Assuming that the calculation related to each state takes $10^{-6}$ seconds, a full calculation of the system will require *over ten years of computer time*. Separation of variables reduces the complexity to a mere 30 x 3 = 90 requiring a negligible amount of computing time.

In Systems Engineering, the separation of variables is done by assuming that components are independent of each other. For example, assume that two components are in series. If they are independent of each other, then the availability of the system is simply $A_1(t) x A_2(t)$ , the product of the availabilities of the two components. Thus, the availability of each component is calculated separately, then the independent functions are combined. This turns a single, two-dimensional calculation into two 1-dimensional calculations.


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Is this simplification justified? It depends on the reality of the system. In a car, the engine and the gearbox are in series. If the gearbox fails the engine is made passive and vice versa. This means that events in the life history of one of the components have an effect on the other. Thus, the components are not independent of each other. In fact, in most systems, components are not independent. Yet, in analytical methods, of necessity, most components must be treated as independent.

Even if we accept the simplification to one-dimensional problems, mathematical complexity still presents a serious barrier to accurate portrayal of the behavior of the system. So, additional approximations are added such as assuming constant failure and repair rates, or assuming immediate repair.

One may assume that components are independent of each other. But try a simple test. Take any system you are familiar with and start describing its operations. Very soon you will reach the conclusion that the elements of the system are not independent. For example, consider a set of jet engines. It seems that each engine lives its life without interaction with any other engine. It flies, it fails, it is repaired, maintained, inspected, etc., but are the jet engines really independent of each other? When engines go into periodic inspection following a fixed number of operating hours, there may be a queue formed for inspection depending on the number of test facilities available and the number of engines waiting for inspection. Thus, the time each engine waits in the queue, which has a direct bearing on its availability, depends on the number of flight hours undertaken by other engines. So, the inspection process creates a dependency between the engines and the jet engines are not really independent of each other.

If some components have many failures, they may exhaust the existing budget and newly failed components may have to wait longer for repair. Therefore, a budget for spare parts can create dependency as well.

Other common and misleading assumptions are:

- assuming that the system is serial;
- assuming constant failure and repair rates;
- assuming immediate repair when preventive maintenance is considered and treating only a single event in these cases;
- assuming that some classes of states never occur.

These assumptions allow analytical methods to be used since, in most cases, they are sufficient to allow something to be calculated, and nobody will ever check their accuracy.

## Summary

Analytical methods are profoundly limited by the mathematical complexity and the dimensionality of systems. In order to use such methods, one is forced to introduce simplifications and approximations. These simplifications, in turn, increase the distance between the predictions and the real behavior of the system. They force ignoring interrelations between events and components of the system, and performance-related aspects of the system. The quality and range of predictions are seriously impaired. This costs big money.

Many of the system features that are neglected due to mathematical limitations simply force the neglect of critical questions. For example, a steady-state availability gives you a single value for the availability, but time-dependent availability may show a point in time at which the availability drops. Just before that point, preventive maintenance and/or inspection could be very useful.

## The Role of SPAR™

SPAR™ is a software application that provides the tools to make such elaborate descriptions of systems and solves the myriad problems described in this paper. The user need not be an expert in MonteCarlo methods. All the knowledge required in the MC method is encapsulated in SPAR™ in such a way to be useful for the most complex and integrated problems of Systems Engineering.

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